EFFECT OF A COATING ON THE ELECTRICAL AND THERMAL RESISTANCE OF A PLANE SINGLE CONTACT FOR A TWO-LAYER HALF-SPACE

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We consider the effect of a coating on the thermal and electrical resistance of a contact spot for a two-layer half-space. An analytical solution is presented. On the basis of the latter and using an approximation, we obtain rather accurate relations that can be used in calculations in designing contact conjugations with coatings.

Calculation of the electrical and thermal resistance of coated bodies is of great practical interest. In electrical contacts, coatings made of noble metals are widely used to decrease contact resistance and friction. In a number of problems, for example, when the heat flux through an oxidized interface or a coated surface is calculated, it is necessary to know the thermal resistance of such a contact. The pressing nature of the problem has drawn the attention of many researchers.

In [1] the influence of the coating of contacting surfaces on the contact electrical resistance is investigated. It is suggested that upper and lower estimates of the contact electrical resistance be made by subdividing the contraction volume by the current tubes and the equipotential surface, respectively. A deficiency of this calculation is insufficient accuracy. In [2, 3] the electrical resistance of point contacts was calculated for multilayer media. However, in these works solutions were obtained by numerical methods, and the results were presented in the form of graphs for, as a rule, separate values of the thicknesses of the coatings and their conductivities.

In [4] the effect of a surface coating on the thermal resistance of an isothermal contact spot on a semiinfinite body is considered. The final solution is presented in the form of a series, an analysis of which made it possible to obtain simple relations for thin and thick coatings. Subsequently, results were obtained for an arbitrary density of an axisymmetric heat flux in a steady state and over short time intervals [5]. The solution was obtained in the form of rather complex series.

Thus, the above brief review shows that despite substantial progress, an engineering procedure for calculating the electrical and thermal resistance for coated bodies is lacking.

The aim of the present work was to obtain analytic relations suitable for engineering calculations of the electrical and thermal resistance of a plane circular contact for a two-layer conducting half-space.

Let us calculate the electrical resistance of a circular contact of radius a for a half-space with electrical conductivity γ_2 having a coating of thickness Δ and electrical conductivity γ_1.

The solution of this problem presupposes that the distribution of the potential is known for the circular contact for a homogeneous conducting half-space. Therefore we will solve an auxiliary problem, i.e., an electrical analog. It is necessary to find the distribution of the potential of a charged conducting oblate ellipsoid of revolution placed in a homogeneous dielectric medium with absolute dielectric permeability ε_a (Fig. 1).

Let us write the equation of an ellipsoid in a Cartesian coordinate system [6]:

$$\frac{X^2 + Y^2}{c^2 (1 + \sigma^2)} + \frac{Z^2}{c^2 \sigma^2} = 1 \tag{1}$$

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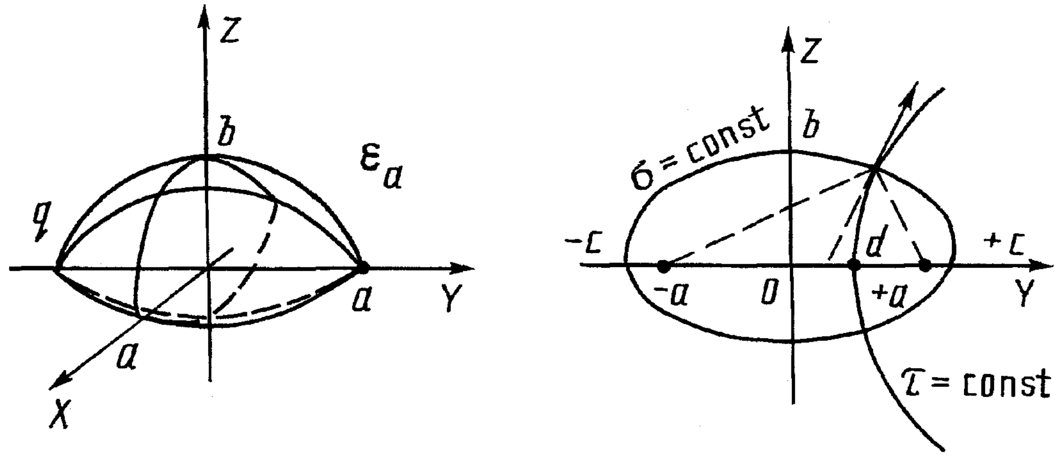


Fig. 1. Computational scheme for an auxiliary problem on the distribution of the potential of an oblate ellipsoid of revolution placed in a homogeneous medium.

and give an equation for one-sheet hyperboloids of rotation that are orthogonal to the ellipsois:

$$\frac{X^2 + Y^2}{c^2 (1 - \tau^2)} - \frac{Z^2}{c^2 \tau^2} = 1 . \tag{2}$$

The equation for the half-plane passing through the OZ axis is

$$Y = X \tan \Theta . \tag{3}$$

The dimensions of the ellipsoid are the following:

$$c = a \sqrt{1 + \sigma^2} , \quad b = c \sigma . \tag{4}$$

To solve the problem, we use the coordinate system of an oblate ellipsoid of revolution σ, τ, φ. Let us assume that $\sigma \geq 0$ and $-1 \leq \tau \leq 1$. In this system, we write the Laplace equation for the potential φ of the ellipsoid:

$$\nabla^2 \Phi = \frac{1}{a^2 (\sigma^2 + \tau^2)} \left\{ \frac{\partial}{\partial \sigma} \left[(1 + \sigma^2) \frac{\partial \varphi}{\partial \sigma} \right] + \frac{\partial}{\partial \tau} \left[(1 - \tau^2) \frac{\partial \varphi}{\partial \tau} \right] + \right.$$

$$\left. + \frac{\sigma^2 + \tau^2}{(1 + \sigma^2)(1 - \tau^2)} \frac{\partial^2 \varphi}{\partial \varphi^2} \right\} = 0 . \tag{5}$$

In the ellipsoidal coordinate system the potential of points of the field will depend only on the coordinate σ:

$$\varphi = \varphi (\sigma) , \tag{6}$$

and then the Laplace equation will have the form

$$\frac{1}{a^3 (\sigma^2 + \tau^2)} \frac{d}{d\sigma} \left[a (1 + \sigma^2) \frac{d\varphi}{d\sigma} \right] = 0 . \tag{7}$$

The solution of this differential equation yields

$$\frac{d\varphi}{d\sigma} = \frac{c_1}{a(1+\sigma^2)}; \quad \varphi = \frac{c_1}{a}\arctan\sigma + c_2. \tag{8}$$

We will assume that the potential of an infinitely distant point is equal to zero: $\sigma \to \infty$, $\varphi \to 0$. Then we obtain

$$\frac{c_1}{a}\frac{\pi}{2} + c_2 = 0; \quad c_2 = -\frac{c_1\pi}{2a}. \tag{9}$$

Let us determine the electric field strength. Since

$$\mathbf{E} = -\operatorname{grad}\varphi,$$

then in the system of coordinates σ, τ, θ we will have

$$\mathbf{E} = -\nabla\varphi = -\frac{\sqrt{1+\sigma^2}}{a\sqrt{\sigma^2+\tau^2}}\frac{\partial\varphi}{d\sigma}\cdot\mathbf{1}_\sigma.$$

Using the results (8) and (9), we obtain

$$\mathbf{E} = -\frac{c_1\sqrt{1+\sigma^2}}{a(1+\sigma^2)\,a\sqrt{\sigma^2+\tau^2}}\cdot\mathbf{1}_\sigma$$

Reducing the minor axis of the ellipsoid to zero, we pass in the limit to a charged circle for which $\sigma \to 0$:

$$\mathbf{E} = -\frac{c_1}{a^2\tau}\cdot\mathbf{1}_\sigma.$$

Converting to the Cartesian coordinate system and using Eq. (4), we obtain

$$\mathbf{E} = -\frac{c_1}{a^2\sqrt{\left(1-\frac{X^2+Y^2}{a^2}\right)}}\cdot\mathbf{1}_k.$$

Using Gauss' theorem, we shall convert to a polar coordinate system in the XOY plane:

$$\Phi_E = -\frac{c_1}{a}2\int_0^a\frac{2\pi r dr}{\sqrt{a^2-r^2}} = \frac{q}{\varepsilon_a}.$$

Having integrated this equation, we find

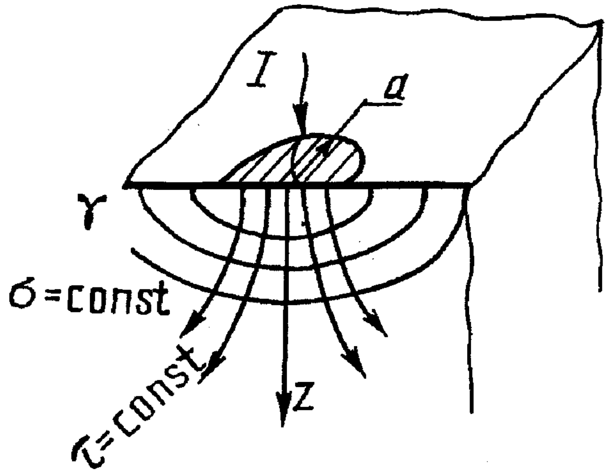
$$c_1 = \frac{q}{4\pi\varepsilon_{a1}}.$$

Thus, for the charged circle the distribution of the potential along the OZ axis in the ellipsoidal coordinate system is expressed by the relation

$$\varphi = \frac{q}{8\varepsilon_{a1}a} - \frac{q}{4\pi\varepsilon_{a1}a}\arctan\sigma.$$

Converting to the Cartesian coordinate system, we have

$$\varphi(Z) = \frac{q}{8\varepsilon_{a1}a} - \frac{q}{4\pi\varepsilon_{a1}a}\arctan\frac{z}{a}. \tag{10}$$



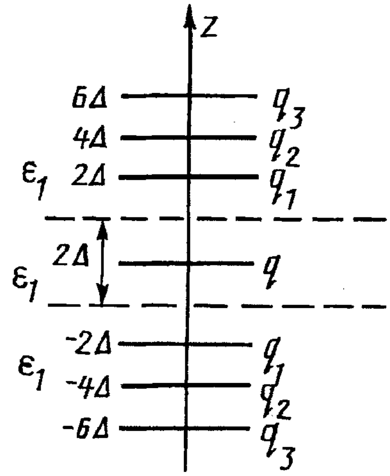


Fig. 2. Pattern of the field for a circular electrical contact in a homogeneous medium.

Fig. 3. Scheme of replacement for a given problem.

Passing to the problem of a circular electrical contact and a homogeneous medium, we will have the picture of the field depicted in Fig. 2 and the following distribution of the potential along the OZ axis:

$$\varphi(Z) = \frac{I}{4\gamma a} - \frac{I}{2\pi\gamma a}\arctan\frac{z}{a}. \tag{11}$$

In particular, for the potential of the electrode and its contact resistance we obtain

$$\varphi(0) = \frac{I}{4\gamma a}, \quad R = \frac{1}{4\gamma a}.$$

This result coincides with a formula that is well known in the theory of electrical contacts [7].

Let us apply formula (10) to the solution of the initial problem, using again the method of electrical analogy.

Suppose it is required that the potential be determined for a charged conducting circle located in a dielectric layer with dielectric permeability ε_1. The remaining medium is homogeneous and has dielectric permeability ε_2. We solve the problem by the method of multiple mirror images using the solution of Searle's problem for the mirror image for the interface between two dielectrics. Passing to the scheme of replacement (see Fig. 3) for the given problem, we will have an infinite sequence of charged disks, where

$$q_1 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q, \quad q_2 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q_1 =$$

$$= \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)^2 q, \ \ldots, \ q_i = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)^i q, \ \ldots$$

The position of the charged disks is determined by the coordinate z and is equal to $|z_1| = 2\Delta$, $|z_2| = 4\Delta$, ..., $|z_i| = 2i\Delta$. Let us determine the potential of the circle located at the point $z = 0$. From the principle of superposition we have

$$\varphi(0) = \frac{q}{8\varepsilon_{a1} a} + 2\left(\frac{q_1}{8\varepsilon_{a1} a} - \frac{q_1}{4\pi\varepsilon_{a1} a}\arctan\frac{2\Delta}{a}\right) + \ldots +$$

$$+ 2\left(\frac{q_i}{8\varepsilon_{a1} a} - \frac{q_i}{4\pi\varepsilon_{a1} a}\arctan\frac{2i\Delta}{a}\right) + \ldots$$

TABLE 1. Comparison of the Values of the Functions f_1 and f_2

t	0	0.001	0.01	0.1	1	2	3	4	5	6	7	8	10	100	$t \to \infty$
f_1	0	0.001	0.01	0.0997	0.785	1.107	1.249	1.326	1.373	1.406	1.429	1.446	1.471	1.561	$\pi/2$
f_2	0	0.001	0.01	0.0969	0.740	1.131	1.338	1.448	1.506	1.536	1.552	1.561	1.568	1.578	$\pi/2$
δ, %	0	0	0	2.8	5.7	2.1	7.1	9.2	9.6	9.3	8.4	7.9	6.6	0.6	0

Denoting the ratio $(\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2) = k$ and grouping the terms, we obtain

$$\varphi(0) = \frac{q}{8\varepsilon_{a1}a} + \sum_{i=1}^{\infty} \frac{k^i q}{4\varepsilon_{a1}a} - \sum_{i=1}^{\infty} \frac{k^i q}{2\pi\varepsilon_{a1}a} \arctan\left(\frac{2i\Delta}{a}\right).$$

Taking into account that

$$\sum_{i=1}^{\infty} k^i = \sum_{i=0}^{\infty} k^i - 1 = \frac{\varepsilon_1 - \varepsilon_2}{2\varepsilon_2},$$

we have

$$\varphi(0) = \frac{q}{4\varepsilon_{a1}a}\left(\frac{1}{2} + \frac{\varepsilon_{a1} - \varepsilon_{a2}}{2\varepsilon_{a2}}\right) - \frac{q}{2\pi\varepsilon_{a1}a} \sum_{i=1}^{\infty} k^i \arctan\left(2i\frac{\Delta}{a}\right).$$

Finally we obtain

$$\varphi(0) = \frac{q}{8\varepsilon_{a2}a} - \frac{q}{2\pi\varepsilon_{a1}a} \sum_{i=1}^{\infty} \left(\frac{\varepsilon_{a1} - \varepsilon_{a2}}{\varepsilon_{a1} + \varepsilon_{a2}}\right)^i \arctan\left(2i\frac{\Delta}{a}\right).$$

Thus, the electric resistance of the contact for the original problem is

$$R = \frac{1}{4\gamma_2 a} - \frac{1}{\pi\gamma_1 a} \sum_{i=1}^{\infty} \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^i \arctan\left(2i\frac{\Delta}{a}\right). \tag{12}$$

The above relations can be rewritten in the form

$$\frac{R_c}{R_0} = 1 - \frac{4}{\pi}\frac{1-k}{1+k} \sum_{i=1}^{\infty} k^i \arctan\left(i\frac{2\Delta}{a}\right). \tag{13}$$

Let us extrapolate the function $f_1 = \arctan t$, where $t = i\cdot(2\Delta/a)$, by the function $f_2 = (\pi/2)(1-\exp(-ct))$ in order that the derivatives of both functions be the same at zero

$$\left.\frac{1}{1+t^2}\right|_{t=0} = \frac{\pi}{2} c \exp(-ct)\Big|_{t=0}.$$

Hence we obtain $c = 2/\pi$.

A comparison of these two functions is given in Table 1, from which it is seen that the maximum error of approximation does not exceed 9.6%. With allowance for the approximation, formula (11) can be transformed to

$$\sum_{i=1}^{\infty} k^i \arctan\left(i\frac{2\Delta}{a}\right) \simeq \sum_{i=1}^{\infty} k^i \frac{\pi}{2}\left(1 - \exp\left(-\frac{4}{\pi}\frac{\Delta i}{a}\right)\right) =$$

$$= \frac{\pi}{2} \frac{k}{1-k} - \frac{\pi}{2} \frac{k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}.$$

Then, incorporating formula (12) we obtain

$$\frac{R_c}{R_0} = 1 - \frac{4}{\pi}\frac{1-k}{1+k}\left(\frac{\pi}{2}\frac{k}{1-k} - \frac{\pi}{2}\frac{k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}\right) =$$

$$= \frac{1-k}{1+k} \frac{1 + k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}.$$

So, finally we have

$$\frac{R_c}{R_0} = 1 - \frac{4}{\pi}\frac{1-k}{1+k}\sum_{i=1}^{\infty} k^i \arctan\left(2i\frac{\Delta}{a}\right), \tag{14a}$$

$$\frac{R_c}{R_0} = \frac{1-k}{1+k} \frac{1 + k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k \exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}, \tag{14b}$$

where

$$k = \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}; \quad \frac{1-k}{1+k} = \frac{\gamma_2}{\gamma_1}.$$

Let us analyze these formulas:

a) $\gamma_1 = \gamma_2$ (homogeneous half-space, $\gamma_m = \gamma_0$), $k = 0$, $R_c/R_0 = 1$, i.e., $R_x/R_0 = 1/4\gamma_2 a$, which coincides with Holm's formula [7];

b) $\gamma_1 \neq \gamma_2$, $\Delta \to \infty$ (homogeneous half-space with $\gamma_m = \gamma_c$), $R_c/R_0 = 1$, i.e., $R_c/R_0 = 1$, i.e., $R_c = R_0 = 1/4\gamma_1 a$;

c) $\Delta \to 0$, $R_c/R_0 = 1$;

d) $\gamma_1 \ll \gamma_2$, $\Delta \ll a$.

Neglecting the spreading in the coating, we have

$$R_c = \frac{\Delta}{\gamma_1 \pi a^2} + \frac{1}{4\gamma_2 a} = \frac{1}{4\gamma_2 a}\left(1 + \frac{4}{\pi}\frac{\gamma_2}{\gamma_1}\frac{\Delta}{a}\right),$$

i.e.,

$$\frac{R_c}{R_0} = 1 + \frac{4}{\pi}\frac{\gamma_2}{\gamma_1}\frac{\Delta}{a}. \tag{15}$$

From Eq. (14b) for $\Delta/a \ll 1$, allowing for the fact that $(1-k)/(1+k) = \gamma_2/\gamma_1$, we write

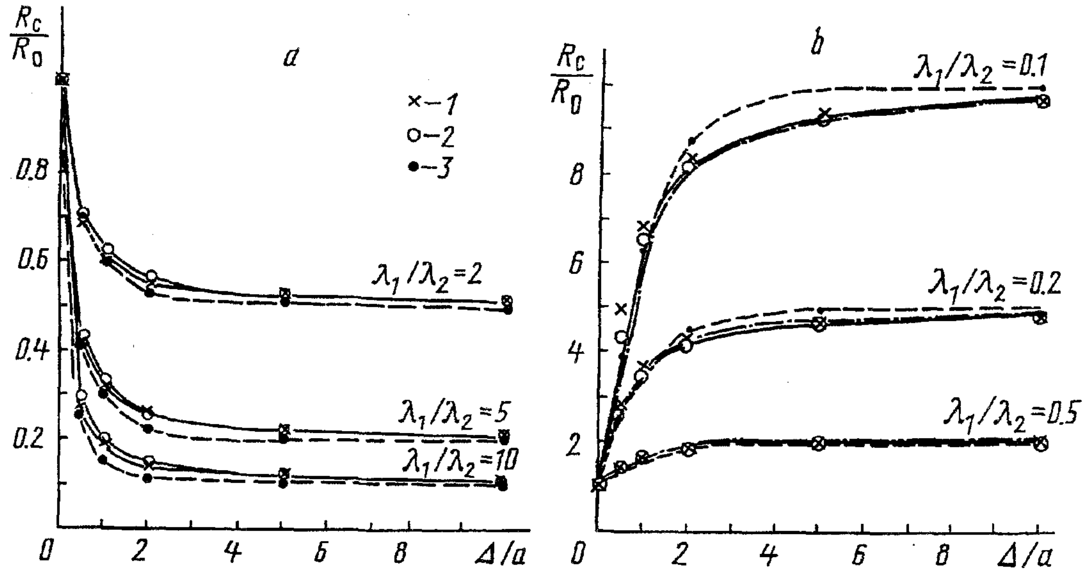


Fig. 4. Comparison of calculations by formulas (17a) and (17b) with the data of [4]: a) for $\lambda_1/\lambda_2 > 1$; b) for $\lambda_1/\lambda_2 < 1$: 1) according to [4]; 2) by formula (17a); 3) by formula (17b).

$$\frac{R_c}{R_0} \simeq \frac{1-k}{1+k} \, \frac{1 + k\left(1 - \frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k\left(1 - \frac{4}{\pi}\frac{\Delta}{a}\right)} = 1 + \frac{4}{\pi}\frac{\gamma_2}{\gamma_1}\frac{\Delta}{a},$$

which coincides with expression (15). Thus, the relations obtained comply with possible particular cases. We shall go over to consideration of the thermal resistance of a circular contact spot for a coated body.

According to the Wiedemann–Franz law, $\gamma \sim \lambda$, and then, taking Eq. (12) into account, we obtain

$$R_T = \frac{1}{4\lambda_2 a} - \frac{1}{\pi\lambda_1 a} \sum_{i=1}^{\infty} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^i \arctan\left(2i\frac{\Delta}{a}\right). \tag{16}$$

Thus, the formulas suggested for calculating the electrical resistance of the contact spot for a coated body are valid for calculating the thermal resistance. Finally we obtain

$$\frac{R_c}{R_0} = 1 - \frac{4}{\pi}\frac{1-k}{1+k} \sum_{i=1}^{\infty} k^i \arctan\left(2i\frac{\Delta}{a}\right), \tag{17a}$$

$$\frac{R_c}{R_0} = \frac{1-k}{1+k} \, \frac{1 + k\exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}{1 - k\exp\left(-\frac{4}{\pi}\frac{\Delta}{a}\right)}, \tag{17b}$$

where

$$k = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}, \quad \frac{1-k}{1+k} = \frac{\lambda_2}{\lambda_1}.$$

An analysis of these formulas is given above. In Fig. 4 dependences of R_c/R_0 on the ratio of the coating thickness Δ and the contact spot radius for different values of λ_1/λ_2 are compared. As is seen from the graphs, expressions

(17a) and (17b) proposed for calculating the thermal resistance of a contact spot for a coated body agree well with solution (22) obtained in [4].

Thus, the solution obtained allows one to take into account the effect of the coating on the electrical and thermal resistance of a contact spot for coated bodies. An approximation of this solution made it possible to obtain simple but rather accurate formulas that can be used in engineering calculations in the design of contact conjugations with coatings.

NOTATION

γ_1, γ_2, electrical conductivity of coating and base; Δ, coating thickness; a, radius of contact spot; ε_a, dielectric permeability of homogeneous medium; q, charge of ellipsoid; c and b, major and minor axes of ellipsoid (Fig. 1); $2a$, focal length of ellipsoid; q, charge of circle (Fig. 3); ε_1 and ε_2, dielectric permeability of the material of coating and base; R_c and R_0, electrical resistance of contact spot for a body with and without a coating; λ_1 and λ_2, thermal conductivity of coating and base.

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